Pergamon

J. Appl. Maths Mechs, Vol. 61, No. 4, pp. 639–651, 1997

0021–8928/97 $24.00+0.00

PII: S0021–8928(97)00082-8

AXIALLY SYMMETRIC DEFORMATION OF THIN SHELLS OF REVOLUTION MADE OF A NON-LINEARLY ELASTIC MATERIAL†

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(*Received* 19 *November* 1995)

Approximate elasticity relations are derived for the axially symmetric deformation of a thin shell of revolution made of a non-linearly elastic material using the three-dimensional equations of the theory of elasticity. The deformations are assumed to be of the order of a small parameter which is proportional to the square root of the dimensionless thickness of the shell. Terms of the second order of smallness with respect to the deformations are retained in the elasticity relations, as a result of which the equations obtained have an error of the order of the dimensionless thickness of the shell, which is customary in the linear theory of shells. The Kirchhoff–Love hypotheses are satisfied only in the first approximation. The axial compression of a shell, assuming that one of the extreme parallels can freely slide along a plane of support, which is perpendicular to the axis of revolution, is considered as an example. A formula is obtained for the limiting load, which physically and geometrically takes account of non-linear effects in the first approximation. © 1997 Elsevier Science Ltd. All rights reserved.

1. THE METRIC OF THE DISTORTED SHELL

Suppose that, prior to deformation, the median surface of a shell of revolution is described by the relations (Fig. 1)

$$r_0 = r_0(s_0), \quad \theta_0 = \theta_0(s_0), \quad r_0' = \cos\theta_0 \tag{1.1}$$

Here s_0 is the length of the arc of the generatrix, r_0 is the distance to the axis of revolution, θ_0 is the angle between the normal to the median surface and the axis of rotation, and a prime denotes differentiation with respect to s_0.

Prior to deformation, the principal radii of curvature of the median surface are determined by the expressions

$$\frac{1}{R_{10}} = \theta_0', \quad \frac{1}{R_{20}} = \frac{\sin\theta_0}{r_0} \tag{1.2}$$

After deformation, we will denote the same quantities by s, r, θ, R_1, R_2, respectively, and formulae analogous to (1.1) and (1.2) hold.

Stretching deformations of an element of the median surface ε_1, ε_2 and the changes in its curvature $\varkappa_1$, $\varkappa_2$ have the form

$$\begin{gathered}\varepsilon_1 = s' - 1, \quad \varepsilon_2 = \frac{r}{r_0} - 1, \quad r' = (1+\varepsilon_1)\cos\theta \\ \varkappa_1 = \frac{1}{R_1} - \frac{1}{R_{10}} = \frac{\theta'}{1+\varepsilon_1} - \theta_0', \quad \varkappa_2 = \frac{1}{R_2} - \frac{1}{R_{20}} = \frac{\sin\theta}{r} - \frac{\sin\theta_0}{r_0}\end{gathered} \tag{1.3}$$

Prior to deformation, we introduce a (material) orthogonal system of curvilinear coordinates $q_1 = s_0$, $q_2 = \varphi$, $q_3 = z$, where φ is the angle in the peripheral direction and z is the distance to the median surface. The square of the distance between infinitely close points is equal to

$$(d\mathbf{R}^0)^2 = H_i^2 dq_i^2 \equiv g_{ij}^0 dq_i dq_j, \quad H_1 = 1 - z\theta_0', \quad H_2 = r_0 - z\sin\theta_0, \quad H_3 = 1 \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 61, No. 4, pp. 660–673, 1997.

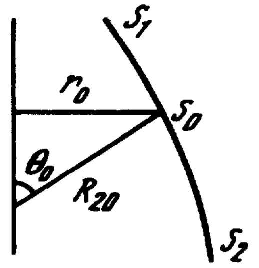

Fig. 1.

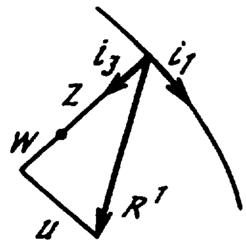

Fig. 2.

where H_i are Lamé coefficients and g_{ij}^0 are the covariant components of the metric tensor prior to deformation.

In order to describe the position of a point (s_0, φ, z) after deformation we will use a moving Cartesian system of coordinates with the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ connected with the deformed median surface (Fig. 2). After deformation, the point (s_0, φ, 0) transfers to the origin of coordinates while, after deformation, the position of the point (s_0, φ, z) is described by the vector

$$\mathbf{R} = \mathbf{R}^0 + \mathbf{R}^1, \quad \mathbf{R}^1 = \mathbf{i}_1 u + \mathbf{i}_3 (z + w) \tag{1.5}$$

The functions $u(s_0, z)$ and $w(s_0, z)$ describe the shear strain and the elongation of a normal to the median surface, respectively. In the case when the Kirchhoff–Love hypothesis is assumed, $u(s_0, z) = w(s_0, z) \equiv 0$.

After deformation, the covariant components of the metric tensor $g_{ij} = \mathbf{R}_i\mathbf{R}_j$, where $\mathbf{R}_i = \partial\mathbf{R}/\partial q$. The formulae $\partial\mathbf{R}_i/\partial q_j = \Gamma_{ij}^k\mathbf{R}_k$ hold, and it is convenient to use these to calculate the Christoffel symbols Γ_{ij}^k.

The components ε_{ij} of the Cauchy–Green strain tensor $\mathbf{E}$ are found using the formulae

$$g_{ij} - g_{ij}^0 = 2H_iH_j\varepsilon_{ij} \tag{1.6}$$

In the case of axisymmetrical deformation, $\varepsilon_{12} = \varepsilon_{23} = 0$.

Expansions of the quantities ε_{ij} and Γ_{ij}^k in power series with respect to a small parameter μ, which is proportional to the square root of the reduced thickness of the shell, are required below. Here, we shall assume that, in the case of the deformed state of the shell being considered, the estimates

$$\begin{gathered} z \sim \mu^2, \quad \{\varepsilon_i, \varepsilon_{ii}\} \sim \mu, \quad \varepsilon_{13} \sim \mu^2, \quad w \sim \mu^3, \quad u \sim \mu^4 \\ \frac{\partial y}{\partial s_0} \sim \frac{y}{\mu}, \quad y = \{\varepsilon_i, \varepsilon_{ij}, \theta, u, w\} \\ \frac{\partial y}{\partial z} \sim \frac{y}{\mu^2}, \quad y = \{\varepsilon_{ij}, u, w\}, \quad \mu^4 = \frac{h^2}{12(1-\nu^2)R^2} \end{gathered} \tag{1.7}$$

hold, where h is the thickness of the shell, R is its characteristic size and ν is Poisson's ratio.

In the formulae presented below the auxiliary parameter $\mu_0 = 1$ fixes the order of the terms: terms with a factor μ_0^k are of the order of μ^k (only non-zero quantities are given)

$$\begin{aligned} \varepsilon_{11} &= \mu_0(\varepsilon_1 + z\theta') + \mu_0^2 \quad (\varepsilon_1 + z\theta')^2/2 + w\theta' - z\theta_0') + O(\mu^3) \\ \varepsilon_{13} &= \mu_0^2(u_z + w')/2 + O(\mu^3) \\ \varepsilon_{22} &= \mu_0\varepsilon_2 + \mu_0^2(\varepsilon_2^2/2 + z\varkappa_2^0) + O(\mu^3), \quad \varkappa_2^0 = (\sin\theta - \sin\theta_0)/r_0 \\ \varepsilon_{33} &= \mu_0 w_z + \mu_0^2 w_z^2/2 + O(\mu^3) \end{aligned} \tag{1.8}$$

$$\begin{aligned} \Gamma_{11}^1 &= \varepsilon_1' + z\theta'' + O(\mu), \quad \Gamma_{11}^3 = -\mu_0^{-1}\theta' + O(1) \\ \Gamma_{13}^1 &= \mu_0^{-1}\theta' + O(1), \quad \Gamma_{13}^3 = w_z' + O(\mu) \end{aligned} \tag{1.9}$$

$$\Gamma_{22}^1 = -r_0(r_0' + r_0\varepsilon_2') + O(\mu), \quad \Gamma_{22}^3 = -r_0 \sin\theta + O(\mu)$$

$$\Gamma_{33}^1 = u_{zz} + O(\mu), \quad \Gamma_{33}^3 = \mu_0^{-1} w_{zz} + O(1)$$

We now introduce the notation

$$\mathbf{k_j} = \frac{\mathbf{R_j}}{\sqrt{g_{jj}}}, \quad \gamma_{jn} = \mathbf{k_j i_n} \tag{1.10}$$

Then

$$\gamma_{ii} = 1 + O(\mu^4), \quad \gamma_{13} = \mu_0^2 w' + O(\mu^3), \quad \gamma_{31} = \mu_0^2 u_z + O(\mu^3) \tag{1.11}$$

2. EQUILIBRIUM EQUATIONS AND ELASTICITY RELATIONS

The equilibrium equations are written in the form [1, 2]

$$\frac{\partial \sigma_{ij}^\circ}{\partial q_i} + \Gamma_{ik}^j \sigma_{ik}^\circ + F_j^\circ = 0, \quad \sigma_{ij}^\circ = V \frac{\sigma_{ij}^*}{H_i H_j}, \quad F_j^\circ = \frac{V F_j^*}{H_j(1+E_j)} \tag{2.1}$$

$$\left(V = H_1 H_2 H_3, \quad E_j = \sqrt{1+2\varepsilon_{jj}} - 1\right)$$

where

$$\sigma_{ij} = \frac{S_i(1+E_j)}{S_i^*} \sigma_{ij}^*, \quad \frac{S_1^*}{S_1} = \sqrt{(1+2\varepsilon_{22})(1+2\varepsilon_{33}) - \varepsilon_{23}^2} \quad (1\,2\,3) \tag{2.2}$$

Here, $\boldsymbol{\sigma}_\mathbf{i} = \sigma_{ij}\mathbf{k_j}$ are the actual stresses in the body after deformation, σ_{ij}^* are the components of the stress energy tensor and $\mathbf{F} = F_j^* \mathbf{k_j}$ is the intensity of the external load per unit volume prior to deformation, which is distributed throughout the volume.

In the case of axisymmetrical deformation of a shell of revolution, one of the equations of (2.1) is identically satisfied.

The boundary conditions on the lateral surfaces have the form

$$\boldsymbol{\sigma}_\mathbf{3} = \mathbf{p}^\pm \equiv p_1^\pm \mathbf{k_1} + p_3^\pm \mathbf{k_3}, \quad \boldsymbol{\sigma}_\mathbf{3} = \sigma_{31}\mathbf{k_1} + \sigma_{33}\mathbf{k_3} \quad \text{when} \quad z = \pm h_2 (h_2 = h/2) \tag{2.3}$$

where $\mathbf{p}^\pm$ is the pressure on the lateral surfaces.

By virtue of formulae (2.2), conditions (2.3) give

$$\sigma_{3j}^* = \frac{S_3^*}{S_3(1+E_j)} p_j^\pm, \quad j = 1,3 \quad \text{when} \quad z = \pm h_2 \tag{2.4}$$

The shells material is assumed to be elastic and isotropic. Suppose that the potential $\Phi(I_1, I_2, I_3)$ is known as a function of the invariants of the strain tensor $\mathbf{E}$ and these invariants are taken as

$$I_1 = \varepsilon_{ii}, \quad I_2 = \varepsilon_{ij}\varepsilon_{ji}, \quad I_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki} \tag{2.5}$$

Then

$$\sigma_{ij}^* = \frac{\partial \Phi}{\partial \varepsilon_{ij}} = \frac{\partial \Phi}{\partial I_1}\frac{\partial I_1}{\partial \varepsilon_{ij}} + \frac{\partial \Phi}{\partial I_2}\frac{\partial I_2}{\partial \varepsilon_{ij}} + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \varepsilon_{ij}} = A_1\delta_{ij} + A_2\varepsilon_{ij} + A_3\varepsilon_{ik}\varepsilon_{kj} \tag{2.6}$$

where δ_{ij} is the Kronecker delta and $A_k = k\partial\Phi/\partial I_k$ $(k = 1, 2, 3)$.

In the case of the five-constant theory of elasticity

$$\Phi = \tfrac{1}{2}\lambda I_1^2 + G I_2 + \alpha_1 I_1^3 + \alpha_2 I_1 I_2 + \alpha_3 I_3 \tag{2.7}$$

we obtain

$$A_1 = \lambda I_1 + 3\alpha_1 I_1^2 + \alpha_2 I_2, \quad A_2 = 2G + 2\alpha_2 I_1, \quad A_3 = 3\alpha_3 \tag{2.8}$$

The potential (2.7) gives the general form of the quadratic dependence of the stresses on the strains. When $\alpha_1 = \alpha_2 = \alpha_3 = 0$, formulae (2.6) give Hooke's law.

Suppose that the third-order elastic moduli α_j in (2.7) do not exceed Young's modulus E in order of magnitude. The stresses are divided by E and expanded in series in powers of μ. Terms of the order of μ and μ^2 are retained.

Formulae (2.6) give

$$\sigma_{jj}^* = A_1 + A_2\varepsilon_{jj} + 3\alpha_3(\varepsilon_{1j}^2 + \varepsilon_{j3}^2), \quad j = 1,3$$
$$\sigma_{13}^* = \varepsilon_{13}(A_2 + 3\alpha_3(\varepsilon_{11} + \varepsilon_{33})), \quad \sigma_{22}^* = A_1 + A_2\varepsilon_{22} + 3\alpha_3\varepsilon_{22}^2 \tag{2.9}$$

By virtue of (1.8) and (2.8), we find

$$A_1 = \mu_0\lambda(\varepsilon_1 + \varepsilon_2 + z\theta' + w_z) + \mu_0^2[\lambda((\varepsilon_1 + z\theta')^2/2 + w\theta' - z\theta_0' + \varepsilon_2^2/2 + z\varkappa_2^0 + w_z^2/2) +$$
$$+3\alpha_1(\varepsilon_1 + \varepsilon_2 - z\theta' + w_z)^2 + \alpha_2((\varepsilon_1 + z\theta')^2 + \varepsilon_2^2 + w_z^2)] + O(E\mu^3) \tag{2.10}$$
$$A_2 = 2G + 2\alpha_2\mu_0(\varepsilon_1 + \varepsilon_2 + z\theta' + w_z) + O(E\mu^2)$$

The stresses σ_{ij}^* are found as a result of substituting formulae (2.10) and (1.8) into (2.9).

3. ASYMPTOTIC INTEGRATION OF SYSTEM (2.1)

Suppose that the shell is initially solely under the action of a boundary load and that there are no external forces distributed throughout the volume and over the lateral surface, that is

$$F_j^\circ = p_j^\pm = 0, \quad j = 1,3 \tag{3.1}$$

We integrate Eq. (2.1) with respect to z from $-h_2$ to h_2. In addition, the first of the equations of (2.1) is multiplied by z and integrated between the same limits. We then obtain three equilibrium "integral" equations

$$\langle\sigma_{1j}^\circ\rangle + \langle\Gamma_{11}^j\sigma_{11}^\circ\rangle + 2\langle\Gamma_{13}^j\sigma_{13}^\circ\rangle + \langle\Gamma_{22}^j\sigma_{22}^\circ\rangle + \langle\Gamma_{33}^j\sigma_{33}^\circ\rangle = 0, \quad j = 1,3$$
$$\langle z\sigma_{11}^\circ\rangle' - \langle\sigma_{13}^\circ\rangle + \langle z\Gamma_{11}^1\sigma_{11}^\circ\rangle + 2\langle z\Gamma_{13}^1\sigma_{13}^\circ\rangle + \langle z\Gamma_{22}^1\sigma_{22}^\circ\rangle + \langle z\Gamma_{33}^1\sigma_{33}^\circ\rangle = 0 \tag{3.2}$$
$$\left(\langle Y\rangle \equiv \frac{1}{h}\int_{-h_2}^{h_2} Y dz\right)$$

By virtue of estimates (1.7), formulae (2.10) and (2.9) give

$$A_1 \sim E\mu, \quad \{A_2, A_3\} \sim E \tag{3.3}$$

$$\{\sigma_{11}^*, \sigma_{22}^*, \sigma_{33}^*\} = O(E\mu), \quad \sigma_{13}^* = O(E\mu^2) \tag{3.4}$$

We recall that the tilde symbol gives an exact estimate of the order while the symbol O gives an upper estimate of the order.

A comparison of the orders of the terms in Eqs (2.1) and (3.2) enables one to refine estimates (3.4) and to obtain estimates for the integral quantities in (3.2).

$$\langle\sigma_{22}^\circ\rangle \sim E\mu, \quad \{\langle\sigma_{11}^\circ\rangle, \langle\sigma_{13}^\circ\rangle\} \sim E\mu^2, \quad \langle z\sigma_{11}^\circ\rangle \sim E\mu^3 \tag{3.5}$$

The discrepancy between the orders of magnitude of the quantities σ_{11}° and $\langle\sigma_{11}^\circ\rangle$ is due to the fact that the mean value of the quantity σ_{11}° is close to zero.

The reduction in the order of magnitude of the quantity σ°_{33} compared with estimate (3.4) imposes constraints on the deformation of the shell by virtue of formulae (2.9).

We write the following approximate expressions for σ^*_{ij}

$$\sigma^*_{11} = \mu_0\{\lambda(\varepsilon_1+\varepsilon_2+z\theta'+w_z)+2G(\varepsilon_1+z\theta')\}+O(E\mu^2)$$

$$\sigma^*_{13} = \mu_0^2 G(u_z+w')+O(E\mu^3)$$

$$\sigma^*_{22} = \mu_0\{\lambda(\varepsilon_1+\varepsilon_2+z\theta'+w_z)+2G\varepsilon_2\}+O(E\mu^2)$$

$$\begin{aligned}\sigma^*_{33} &= \mu_0\{\lambda(\varepsilon_1+\varepsilon_2+z\theta'+w_z)+2Gw_z\}+\mu_0^2\{\lambda((\varepsilon_1+z\theta')^2/2+w\theta'-z\theta_0'+\varepsilon_2^2/2+\\ &+z\varkappa_2^0+w_z^2/2)+Gw_z^2+3\alpha_3 w_z^2+2\alpha_2(\varepsilon_1+\varepsilon_2+z\theta'+w_z)w_z+3\alpha_1(\varepsilon_1+\varepsilon_2+z\theta'+w_z)^2+\\ &+\alpha_2((\varepsilon_1+z\theta')^2+\varepsilon_2^2+w_z^2)\}+O(E\mu^3)\end{aligned} \quad (3.6)$$

By virtue of the estimates $\sigma^\circ_{33} \sim E\mu^2$ and $\langle\sigma^\circ_{11}\rangle \sim E\mu^2$, in the zeroth approximation we obtain

$$\lambda(\varepsilon_1+\varepsilon_2+z\theta'+w_z)+2Gw_z=0$$
$$\langle\lambda(\varepsilon_1+\varepsilon_2+z\theta'+w_z)+2G(\varepsilon_1+z\theta')\rangle=0$$

whence the relations

$$\begin{aligned}&\varepsilon_1=-\nu\varepsilon_2+O(\mu^2),\quad \sigma^*_{11}=\frac{E}{1-\nu^2}z\theta'+O(E\mu^2)\\ &w_z=-\nu\varepsilon_2-\frac{\nu}{1-\nu}z\theta'+O(\mu^2),\quad w=-\nu\varepsilon_2 z-\frac{\nu}{2(1-\nu)}z^2\theta'+O(\mu^2)\end{aligned} \quad (3.7)$$

follow.

The asymptotically principal terms of Eqs (2.1) give

$$\frac{\partial(r_0\sigma^*_{11})}{\partial s_0}+r_0\frac{\partial\sigma^*_{13}}{\partial z}=O(E\mu),\quad \frac{\partial\sigma^*_{33}}{\partial z}-\theta'\sigma^*_{11}=O(E\mu) \quad (3.8)$$

Integrating Eqs (3.8) with respect to z and taking account of expression (3.7) for σ^*_{11} and the boundary conditions $\sigma^*_{13}=\sigma^*_{33}=0$ when $z=\pm h_2$, we find

$$\sigma^*_{13}=\frac{E(h_2^2-z^2)}{2(1-\nu^2)r_0}(r_0\theta')'+O(E\mu^3),\quad \sigma^*_{33}=-\frac{E(h_2^2-z^2)}{2(1-\nu^2)}\theta'^2+O(E\mu^3) \quad (3.9)$$

By using expression (3.6) for σ^*_{33}, we find an expression for w_z which is more exact than (3.7). Here, when calculating terms of the order of μ^2, we have used the approximate formulae (3.7). We obtain

$$\begin{aligned}&w_z=-\mu_0\left(\nu\varepsilon_2+\frac{\nu}{1-\nu}z\theta'\right)-\frac{\mu_0^2}{\lambda+2G}\{\lambda(\varepsilon_1+\nu\varepsilon_2)+b_1\varepsilon_2^2+\\ &+z[\lambda(-\theta_0'+\varkappa_2^0)+b_2\varepsilon_2\theta']+b_3 z^2\theta'^2-\sigma^*_{33}\}+O(E\mu^3)\\ &b_1=\frac{E\nu}{2(1-2\nu)}+\alpha_1 3(1-2\nu)^2+\alpha_2(1-2\nu+6\nu^2)+3\nu^2\alpha_3\\ &b_2=-\frac{E\nu^2}{(1+\nu)(1-2\nu)}+\alpha_1\frac{6(1-2\nu)^2}{1-\nu}-\alpha_2\frac{6\nu(1-2\nu)}{1-\nu}+\alpha_3\frac{6\nu^2}{1-\nu}\end{aligned} \quad (3.10)$$

$$b_3 = \frac{E\nu}{2(1+\nu)(1-2\nu)} + \alpha_1 \frac{3(1-2\nu)^2}{(1-\nu)^2} + \alpha_2 \frac{1-4\nu+6\nu^2}{(1-\nu)^2} + \alpha_3 \frac{3\nu^2}{(1-\nu)^2}$$

It is convenient to find the stresses σ_{11}^* and σ_{22}^* from the relations

$$\sigma_{kk}^* = A_2(\varepsilon_{kk} - \varepsilon_{33}) + 3\alpha_3(\varepsilon_{kk}^2 - \varepsilon_{33}^2) + \sigma_{33}^*, \quad k = 1,2 \tag{3.11}$$

which follow from (2.9).

Substituting formulae (1.8), (2.10), (3.7), (3.9) and (3.10) into (3.11), we obtain

$$\begin{aligned} \frac{\sigma_{11}^*}{E} &= \mu_0 \frac{z\theta'}{1-\nu^2} + \mu_0^2 \frac{1}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2 + \nu z \varkappa_2^0 - z\theta_0') + \\ &+ \mu_0^2\{b_{11}\varepsilon_2^2 + b_{12}\varepsilon_2 z\theta' + b_{13}(z\theta')^2 + b_{14}\theta'^2(h_2^2 - z^2)\} + O(\mu^3) \\ \frac{\sigma_{22}^*}{E} &= \mu_0\left(\varepsilon_2 + \frac{\nu z\theta'}{1-\nu^2}\right) + \mu_0^2 \frac{\nu}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2 + z\varkappa_2^0 - \nu z\theta_0') + \\ &+ \mu_0^2\{b_{21}\varepsilon_2^2 + b_{22}\varepsilon_2 z\theta' + b_{23}(z\theta')^2 + b_{24}\theta'^2(h_2^2 - z^2)\} + O(\mu^3) \end{aligned} \tag{3.12}$$

where b_{ij} are linear functions of the dimensionless quantities $\alpha_j^0 = \alpha_j/E$ (j = 1, 2, 3) with coefficients which depend on ν.

4. TWO-DIMENSIONAL EQUILIBRIUM EQUATIONS

The equilibrium equations of a curvilinear parallelepiped, which (before deformation) occupies the domain s_0, $s_0 + ds_0$; φ, $\varphi + d\varphi$; $-h_2 \leqslant z \leqslant h_2$, have the form (after deformation) in projections on the tangent and the normal to the median surface

$$\begin{aligned} &(r_0T_1)' - T_2\cos\theta + r_0\theta' Q_1 + r_0 p_1 = 0 \\ &(r_0Q_1)' - T_2\sin\theta - r_0\theta' T_1 + r_0 p_3 = 0 \\ &(r_0M_1)' - M_2\cos\theta - r_0(1+\varepsilon_1)Q_1 + r_0 m_1 = 0 \end{aligned} \tag{4.1}$$

where the projections of the forces and the moments T_j, Q_1, M_j have been divided by the unit of length of the median surface prior to deformation and are determined starting from the formulae

$$\begin{aligned} &T_1\mathbf{i}_1 + Q_1\mathbf{i}_3 = \left\langle H_2 \frac{S_1^*}{S_1}\boldsymbol{\sigma}_1 \right\rangle h, \quad M_1\mathbf{i}_2 = \left\langle H_2 \frac{S_1^*}{S_1}\mathbf{R}^1 \times \boldsymbol{\sigma}_1 \right\rangle h \\ &T_2\mathbf{i}_2 = \left\langle H_1 \frac{S_2^*}{S_2}\boldsymbol{\sigma}_2 \right\rangle h, \quad M_2\mathbf{i}_1 = -\left\langle H_1 \frac{S_2^*}{S_2}\mathbf{R}^1 \times \boldsymbol{\sigma}_2 \right\rangle h \end{aligned} \tag{4.2}$$

Here, $\boldsymbol{\sigma}_i = \sigma_{ij}\mathbf{k}_j$ are the actual stresses in the body after deformation and the vector $\mathbf{R}^1$ is given by (1.5).

In (4.1), p_1, p_3, m_1 are the projections of the external forces and moments per unit area of the median surface, before deformation, onto the unit vectors $\mathbf{i}_j$.

The system of equations (4.1), taking (4.2) into account, is exact and equivalent to system (3.2) (if the latter is supplemented with terms which take account of the external load). Inaccuracies arise as a result of replacing the exact elasticity relations (4.2) and values of σ_{ij}^* by approximate ones.

In order to derive approximate elasticity relations, we project relations (4.2) onto the unit vectors $\mathbf{i}_j$, take account of formulae (2.2), (1.10) and (3.26) and integrate with respect to z in accordance with (3.2).

Then

$$
\begin{aligned}
T_1 &= \mu_0^2\{K(\varepsilon_1+\nu\varepsilon_2)+Eb_{11}h\varepsilon_2^2+a_1\theta'^2\}+O(Eh\mu^3)\\
T_2 &= \mu_0 Eh\varepsilon_2+\mu_0^2\{K\nu(\varepsilon_1+\nu\varepsilon_2)+(E+Eb_{21})h\varepsilon_2^2+a_2\theta'^2\}+O(Eh\mu^3)\\
M_1 &= \mu_0^3 D\theta'+\mu_0^4\{D(-\theta_0'+\nu\varkappa_2^0)+a_3\varepsilon_2\theta'\}+O(Eh^2\mu^3)\\
M_2 &= \mu_0^3 D\nu\theta'+\mu_0^4\{D(\varkappa_2^0-\nu\theta_0')+a_4\varepsilon_2\theta'\}+O(Eh^2\mu^3)
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
&K=\frac{Eh}{1-\nu^2},\quad D=\frac{Eh^3}{12(1-\nu^2)},\quad a_1=D+\frac{E(b_{13}+2b_{14})h^3}{12}\\
&a_2=\frac{E(b_{23}+2b_{24})h^3}{12},\quad a_3=\frac{Eb_{12}h^3}{12}-2\nu D,\quad a_4=\frac{Eb_{22}h^3}{12}+\frac{(\nu-3\nu^2)D}{2}
\end{aligned}
\tag{4.4}
$$

The system of equations (4.1), together with the geometric equation

$$
(r_0\varepsilon_2)'=(1+\varepsilon_1)\cos\theta-\cos\theta_0 \tag{4.5}
$$

which follows from formulae (1.3), and relations (4.3) form a close system in the eight unknown functions T_j, M_j, ε_j, Q_1, θ. This system is of the fifth order and T_1, M_1, ε_2, Q_1, θ are the principal unknowns in it, while the functions T_2, M_2, ε_1 are expressed in terms of the principal unknowns using (4.3). We now supplement Eqs (4.1) and (4.5) with the equation

$$
D\theta'=M_1+D(\theta_0'-\nu\varkappa_2^0)-2Dc_2\varepsilon_2\theta' \tag{4.6}
$$

and the relations

$$
T_2=\nu T_1+Eh(\varepsilon_2+c_1\varepsilon_2^2)+Dc_2\theta'^2,\quad \varepsilon_1=-\nu\varepsilon_2,\quad M_2=\nu M_1 \tag{4.7}
$$

which follow from (4.3). The latter two approximate relations ensure the accuracy, which is required later.

The dependence on the non-linear properties of the material manifests itself solely in terms of the coefficients c_1 and c_2 which are equal to

$$
\begin{aligned}
c_1 &= 3/2+3\alpha_1^0(1-2\nu)^3+3\alpha_2^0(1-2\nu)(1+2\nu^2)+3\alpha_3^0(1-2\nu^3)\\
c_2 &= -2\nu+\frac{1+\nu}{1-\nu}[3\alpha_1^0(1-2\nu)^3+\alpha_2^0(1-2\nu)(1-4\nu+6\nu^2)-3\alpha_3^0\nu(1-2\nu+2\nu^2)]
\end{aligned}
\tag{4.8}
$$

The same coefficients occur in the approximate expression for the elastic energy potential

$$
\Pi=\int_{s_1}^{s_2}2\pi r_0\left[Eh\left(\frac{1}{2}\varepsilon_2^2+\frac{c_1}{3}\varepsilon_2^3\right)+D\left(\frac{1}{2}\theta'^2+\theta'(\nu\varkappa_2^0-\theta_0')+c_2\varepsilon_2\theta'^2\right)\right]ds_0 \tag{4.9}
$$

The relative formal error of the resulting system (4.1), (4.5)–(4.7), as well as of expression (4.9), is of the order of magnitude of μ^2 or of the order of magnitude of the dimensionless thickness of the shell. Here, the question is the error in constructing the internal stressed state, and the issue on its interaction with the boundary layer (see [3]) is not considered.

Remark. Formulae (3.12) were obtained under the assumption (3.1), which includes the statement that the shell is not loaded. It can be verified that the order of magnitude of the residual terms $O(\mu)$ and $O(\mu^3)$ in relations (3.8) and (3.9) is preserved when

$$
\overset{\circ}{F}_j=O(\mu),\quad p_j^{\pm}=O(\mu^3),\quad j=1,3 \tag{4.10}
$$

Together with relations (3.8) and (3.9), the terms in formulae (3.12), which have been explicitly written out, also remain unchanged. As a rule, estimates (4.10) are satisfied because stability loss in the linear approximation occurs at far smaller loads

$$F_j^\circ = O(\mu^2), \quad p_j^\pm = O(\mu^4), \quad j = 1,3 \tag{4.11}$$

Estimates (4.10) are violated if the shell is acted upon by a very large external pressure $p_3^\pm \sim \mu^2$, which is identical on both sides of the shell. In this case, the quantity w_z in (3.13) acquires a constant term and formulae (3.16) and (4.3) also change. This case is not considered here.

5. THE PROBLEM OF COMPRESSION BY AN AXIAL FORCE

Suppose that a shell is acted upon by an axial force P applied to its ends, that the edge of the shell $s_0 = s_1$ is clamped in a radial direction and that the edge $s_0 = s_2$ can freely slide along a support plane, which is perpendicular to the axis of revolution. A stressed state, which satisfies estimates (1.7), then occurs in the neighbourhood of the edge $s_0 = s_2$ when $P \sim Eh^2$.

As the characteristic dimension R of the shell, we take $R = r_0(s_2)$ and change to dimensionless variables (with the superscript °) using the formulae

$$\begin{gathered}\{s_0, r_0\} = R\{s_0^\circ, r_0^\circ\}, \quad \{\varepsilon_1, \varepsilon_2\} = \mu\{\varepsilon_1^\circ, \varepsilon_2^\circ\}, \quad \{p_1, p_3\} = Eh\mu^2 R^{-1}\{p_1^\circ, p_3^\circ\} \\ \{T_1, Q_1\} = Eh\mu^2\{T_1^\circ, Q_1^\circ\}, \quad T_2 = Eh\mu T_2^\circ, \quad \{M_1, M_2\} = EhR\mu^3\{M_1^\circ, M_2^\circ\}\end{gathered} \tag{5.1}$$

Substitutions (5.1) are introduced in such a way that the dimensionless quantities are of the order of magnitude of unity in the neighbourhood of the edge $s_0 = s_2$.

We now transfer to the projections V and U of the internal forces in the axial direction and the direction perpendicular to it, respectively

$$T_1^\circ = U\cos\theta + V\sin\theta, \quad Q_1^\circ = U\sin\theta - V\cos\theta \tag{5.2}$$

The degree sign is henceforth omitted.

The system of equations in the unknowns

$$V, U, \varepsilon_2, M_1, \theta \tag{5.3}$$

then takes the form

$$\begin{aligned} &(r_0V)' = 0 \\ &\mu(r_0U)' = \varepsilon_2 + \mu\nu(U\cos\theta + V\sin\theta) + \mu c_1\varepsilon_2^2 + \mu^3 c_2\theta'^2 \\ &\mu(r_0\varepsilon_2)' = (1 - \mu\nu\varepsilon_2)\cos\theta - \cos\theta_0 \\ &\mu(r_0M_1)' = r_0(1 - \mu\nu\varepsilon_2)(U\sin\theta - V\cos\theta) + \mu\nu M_1\cos\theta \\ &\mu\theta' = M_1 + \mu(\theta_0' - \nu\varkappa_2^0) - 2\mu^2 c_2\varepsilon_2\theta' \end{aligned} \tag{5.4}$$

The boundary conditions have the form

$$\begin{aligned} &\varepsilon_2 = 0, \ (\theta = \theta_0 \text{ when } M_1 = 0) \text{ when } s_0 = s_1 \\ &U = M_1 = 0 \text{ when } s_0 = s_2 \end{aligned} \tag{5.5}$$

The first of Eqs (5.4) gives

$$V = \frac{C}{r_0}, \quad P = 2\pi EhR\mu^2 C = \frac{2\pi Eh^2}{\sqrt{12(1-\nu^2)}}C \tag{5.6}$$

We will next seek the limiting value of the force P.

6. ASYMPTOTIC INTEGRATION OF SYSTEM (5.4)

Far from the edges, the shell is deformed and both the zero-moment

$$\theta = \theta_0 + O(C\mu^2), \quad M_1 = O(C\mu^3), \quad U = \frac{C\cos\theta_0}{r_0\sin\theta_0}, \quad \varepsilon_2 = O(C\mu) \tag{6.1}$$

and the principal deformations are concentrated in the neighbourhood of the edge $s_0 = s_2$.

We write system (5.4) in vector form

$$\mu\mathbf{x}' = \mathbf{F}(\mathbf{x}, s_0, C, \mu), \quad \mathbf{x} = \{U, \varepsilon_2, M_1, \theta\} \tag{6.2}$$

carry out a scale expansion $\xi = (s_0 - s_2)/\mu$ and seek its solution in the form

$$\mathbf{x} = \mathbf{x}^0(\xi) + \mu\mathbf{x}^1(\xi) + O(\mu^2), \quad C = C^0 + \mu C^1 + O(\mu^2) \tag{6.3}$$

The accuracy of system (5.4) enables one to construct the two first terms of series (6.3).

Taking account of the fact that

$$r_0(s_0) = 1 + \mu\xi\cos\gamma + O(\mu^2), \quad \gamma = \theta(s_2); \quad \theta_0(s_0) = \gamma + \mu\xi k_1 + O(\mu^2), \quad k_1 = \theta_0'(s_2)$$

and equating the coefficients of μ^0 and μ^1 in system (5.4), we obtain in the zeroth approximation

$$\dot{\mathbf{x}}^0 = \mathbf{F}^0 \equiv \mathbf{F}(\mathbf{x}^0, s_2, C^0, 0) \tag{6.4}$$

or (see also [4, 5])

$$\begin{gathered}\dot{U}^0 = \varepsilon_2^0, \quad \dot{\varepsilon}_2^0 = \cos\theta^0 - \cos\gamma \\ \dot{M}_1^0 = U^0\sin\theta^0 - C^0\cos\theta^0, \quad \dot{\theta}^0 = M_1^0\end{gathered} \tag{6.5}$$

(a dot denotes a derivative with respect to ξ) and in the first approximation

$$\dot{\mathbf{x}}^1 = \mathbf{L}(\xi)\mathbf{x}^1 + \mathbf{g}, \quad \mathbf{L} = \frac{\partial\mathbf{F}^0}{\partial\mathbf{x}^0}, \quad \mathbf{g} = \frac{\partial\mathbf{F}}{\partial s_0}\xi + \frac{\partial\mathbf{F}}{\partial C_0}C^1 + \frac{\partial\mathbf{F}}{\partial\mu}\mu \tag{6.6}$$

or

$$\begin{gathered}\dot{U}^1 = \varepsilon_2^1 + g_1, \quad \dot{\varepsilon}_2^1 = -\theta^1\sin\theta^0 + g_2 \\ \dot{M}_1^1 = U^1\sin\theta^0 + (U^0\cos\theta^0 + C^0\sin\theta^0)\theta^1 + g_3, \quad \dot{\theta}^1 = M_1^1 + g_4\end{gathered} \tag{6.7}$$

where

$$\begin{aligned}g_1 &= -\xi\dot{U}^0\cos\gamma + U^0(\nu\cos\theta^0 - \cos\gamma) + \nu C^0\sin\theta^0 + c_1\varepsilon_2^{0^2} + c_2 M_1^{0^2} \\ g_2 &= -\xi\dot{\varepsilon}_2^0\cos\gamma - \varepsilon_2^0(\nu\cos\theta^0 + \cos\gamma) + k_1\xi\sin\gamma \\ g_3 &= -\nu\varepsilon_2^0(U^0\sin\theta^0 - C^0\cos\theta^0) - C^1\cos\theta^0 + \xi C^0\cos\theta^0\cos\gamma + M_1^0(\nu\cos\theta^0 - \cos\gamma) \\ g_4 &= k_1 - \nu(\sin\theta^0 - \sin\gamma) - 2c_2\varepsilon_2^0 M_1^0\end{aligned} \tag{6.8}$$

The partial derivatives in expression (6.6) for $\mathbf{g} = \{g_1, g_2, g_3, g_4\}$ are calculated for the same values of the arguments as in (6.4).

When $s_0 < s_2$, the solution of system (5.4) changes into the zero-moment solution (6.1) which gives the boundary conditions when $\xi \to -\infty$.

$$U^0 = \frac{C^0\cos\gamma}{\sin\gamma}, \quad \varepsilon_2^0 = M_1^0 = 0, \quad \theta^0 = \gamma \quad \text{when } \xi \to -\infty \tag{6.9}$$

$$U^1 = \frac{C^1 \cos\gamma}{\sin\gamma} - \frac{C^0 (k_1 \xi \sin\gamma + \cos^2\gamma)}{\sin^2\gamma}, \quad \varepsilon_2^1 = M_1^1 = 0, \quad \theta^1 = k_1 \xi \tag{6.10}$$

The boundary conditions (5.5) give

$$U^0 = M_1^0 = 0, \quad U^1 = M_1^1 = 0 \quad \text{when} \quad \xi = 0 \tag{6.11}$$

7. CALCULATION OF THE AXIAL FORCE

System (6.5) has the integral [4]

$$\varepsilon_2^{0^2} - M_1^{0^2} - 2U^0(\cos\theta^0 - \cos\gamma) - 2C^0(\sin\theta^0 - \sin\gamma) = 0 \tag{7.1}$$

in which the arbitrary constant is found by virtue of conditions (6.9). The "shooting" method is used in the numerical solution of the zeroth approximation boundary-value problem (6.5), (6.9), (6.11). The initial conditions

$$U^0 = M_1^0 = 0, \quad \theta^0 = \tau, \quad \varepsilon_2^0 = \tau_1 \quad \text{when} \quad \xi = 0 \tag{7.2}$$

are specified.

It is assumed that the parameter τ is specified, and τ_1 is found from relation (7.1)

$$\tau_1 = \sqrt{2C^0(\sin\tau - \sin\gamma)}\, \mathrm{sign}(\pi/2 - \gamma) \tag{7.3}$$

The constant C^0 is selected from conditions (6.9) and, for these conditions to be satisfied, it is sufficient to require that

$$\|\mathbf{x}^0(\xi) - \mathbf{x}^0(-\infty)\| \to 0 \quad \text{when} \quad \xi \to -\infty, \quad \|\mathbf{x}\| \equiv (x_1^2 + \ldots + x_4^2)^{1/2} \tag{7.4}$$

If relation (7.4) is satisfied, then $C^0 = C^0(\tau)$, $\mathbf{x}^0 = \mathbf{x}^0(\xi, \tau)$. The value of C^0, which corresponds to the limit point in the zeroth approximation, is found from the condition

$$dC^0/d\tau = 0 \tag{7.5}$$

The value of C^1 is found from the condition

$$\int_{-\infty}^{0} (g_1 \varepsilon_2^* - g_2 U^* + g_3 \theta^* - g_4 M_1^*)\, d\xi = 0 \tag{7.6}$$

for the existence of a solution $\mathbf{x}^1$ of system (6.7) which satisfies boundary conditions (6.10) and (6.11). Here, $\mathbf{x}^* = \{U^*, \varepsilon_2^*, M_1^*, \theta^*\}$ satisfies the system $\dot{\mathbf{x}}^* = \mathbf{L}(\xi)\mathbf{x}^*$ and the null boundary conditions ($\mathbf{x}^*(-\infty) = U^*(0) = M^*{}_1(0) = 0$).

When conditions (7.4) are satisfied, we have

$$\mathbf{x}^* = \partial \mathbf{x}^0 / \partial \tau \tag{7.7}$$

which enables us to express C^1 from (7.6) solely in terms of the zeroth approximation $\mathbf{x}^0$.

After some reduction, we obtain

$$C^1 = f_0 + f_k k_1 + f_\nu \nu + f_1 \alpha_1 + f_2 \alpha_2 + f_3 \alpha_3 \tag{7.8}$$

where

$$f_0 = \frac{j_1 + j_2}{2}, \quad f_k = j_3, \quad f_\nu = \frac{j_2 - 5j_4 + 2j_5}{2}$$

$$f_1 = (1 - 2\nu)^3 j_2 + \frac{3(1+\nu)(1-2\nu)^3}{1-\nu} j_4$$

$$f_2 = (1-2\nu)(1+2\nu^2)j_2 + \frac{(1+\nu)(1-2\nu)(1-4\nu+6\nu^2)}{1-\nu} j_4 \tag{7.9}$$

$$f_3 = (1-2\nu^3)j_2 - \frac{3\nu(1+\nu)(1-2\nu+2\nu^2)}{1-\nu} j_4$$

$$j_i = J_i / J_0, \quad i = 1,\ldots,5$$

and the values of J_i depend solely on the angle γ and are determined in the course of the numerical integration of system (6.5) using the formulae

$$J_0 = \frac{\partial}{\partial\tau}\int_{-\infty}^{0}(\sin\theta^0 - \sin\gamma)d\xi, \quad J_1 = \cos\gamma\frac{\partial}{\partial\tau}\int_{-\infty}^{0}\xi((M_1^0)^2 - (\varepsilon_2^0)^2)d\xi$$

$$J_2 = \frac{\partial}{\partial\tau}\int_{-\infty}^{0}(\varepsilon_2^0)^3 d\xi, \quad J_3 = -1 - \sin\gamma\frac{\partial}{\partial\tau}\int_{-\infty}^{0}(U^0(\xi) - U^0(-\infty))\xi d\xi$$

$$J_4 = \frac{\partial}{\partial\tau}\int_{-\infty}^{0}\varepsilon_2^0(M_1^0)^2 d\xi, \quad J_5 = \sin\tau - \sin\gamma \tag{7.10}$$

The value of C^0 and the coefficients in formulae (7.8) and (7.9) are presented below for different values of γ

γ, degree	5	10	20	30	45	60	70	80	85
$C^0 \times 10^4$	–30	–121	–481	–1078	–2402	–4221	–5704	–7439	–8460
$f_0 \times 10^4$	45	124	307	439	336	–323	–1070	–1868	–2069
$f_k \times 10^4$	4330	6104	8554	10362	12330	13637	14192	14741	15654
$f_\nu \times 10^4$	131	366	1007	1751	2807	3430	3376	2718	2024
$j_2 \times 10^4$	0	–6	–65	–260	–989	–2374	–3563	–4562	4570
$j_4 \times 10^4$	0	–1	–6	–24	–85	–172	–211	–183	–119

The case when $\gamma > \pi/2$ reduces to the substitution $\gamma^0 = \pi - \gamma$ and, moreover, it turns out that

$$C^0(\pi-\gamma) = C^0(0), \quad C^1(\pi-\gamma) = -C^1(\gamma) \tag{7.11}$$

When $\gamma \ll 1$, the substitutions

$$\{U^0, \theta^0\} = \gamma\{U^*, \theta^*\}, \ \{M_1^0, \varepsilon_2^0\} = \gamma^{3/2}\{M_1^*, \varepsilon_2^*\}, \ \xi = \gamma^{-1/2}\xi^*, \ C^0 = \gamma^2 C^* \tag{7.12}$$

reduce system (6.4) to a form which does not contain γ (see [5])

$$\frac{t^2U^*}{d\xi^{*2}} = \frac{1}{2}(1-\theta^{*2}), \quad \frac{d^2\theta^*}{d\xi^{*2}} = \theta^* U^* - C^*$$

$$\theta^* \to 1, \quad U^* \to C^* \ \text{when} \ \xi^* \to -\infty \tag{7.13}$$

$$U^* = 0, \quad M_1^* = 0 \ \text{when} \ \xi^* = 0$$

When $\gamma \ll 1$, for the coefficients presented above we have the approximate equalities

$$\{f_0, f_\nu\} = \gamma^{3/2}\{f_0^*, f_\nu^*\}, \quad f_k = \gamma^{1/2} f_k^*, \quad \{j_2, j_4\} = \gamma^{7/2}\{j_2^*, j_4^*\} \tag{7.14}$$

The limit value of the quantity C^* and the numerical values of the coefficients with an asterisk are $C^* = -03963, f_0^* = 0.1768, f_k^* = 1.4602, f_\nu^* = 0.5092, j_2^* = -0.2652, j_4^* = -0.0268$.

It follows from (7.14) that the contribution from the physically non-linear terms (with the factors c_1 and c_2 in (5.4)) is reduced for shallow shells.

8. DISCUSSION OF THE RESULTS

The problem of constructing a linear two-dimensional theory of thin shells starting from the three-dimensional theory of elasticity has been considered in many papers (see [3, 6]).

The approximate elasticity relations (4.3) for the axisymmetrical deformation of a non-linearly elastic shell of a revolution have been obtained here using the three-dimensional theory. Relations (1.7), which are based on an assumption regarding the order of magnitude of the strains $\varepsilon_{ij} = O(\mu)$, are the basis of the derivation. Relations (1.7) have been obtained previously [4, 5] in the case of large supercritical displacements of shells of revolution. The strains $\varepsilon_{ij} = O(\mu)$ are relatively large since, for example, the subcritical strains accompanying the loss of stability of a cylindrical shell during axial compression are of a much lower order of magnitude $\varepsilon_{ij} = O(\mu^2)$. Hence, if we wish to obtain the two-dimensional equations with an error of the order of μ^2, or of the order of h/R in the case of the strains being considered, it is necessary to take account of the non-linear dependence of the stresses on the strains.

The non-linear elasticity relations (4.3), which have an error of the order of μ^2 when $\varepsilon_{ij} = O(\mu)$, have been obtained above for the five-constant elastic material (2.7) under the assumption that $\alpha_j = O(E)$. In the case of strains of the same order of μ, relations (4.3) also hold for an arbitrary non-linearly elastic material under the assumption that coefficients of the fourth and higher orders in expansion (2.7) of the elastic potential Φ in series in the invariants I_k do not substantially exceed E.

In the case of large distortions (of the order of unity), no rigorous derivation of the elasticity relations for shells from the three-dimensional equations of the theory of elasticity can be found in the literature. Elasticity relations are available (see [7, 8], etc.) which have been obtained using the Kirchhoff–Love hypothesis. In the case of small deformations (of the order of μ), a comparison with formulae (4.3) reveals a difference in the non-linear terms.

System (5.4), when $c_1 = c_2 = 0$, has been investigated in [4, 5, 9] using the method of asymptotic integration. A problem of large axisymmetrical supercritical deformations of a shell by an external pressure has been considered in [4, 5]. Here, the shape of a part of the median surface is close to the initial shape while the other part is close to that obtained by a mirror reflection from a plane perpendicular to the axis of revolution. A system, which is equivalent to (6.5) when $C^0 = 0$, has been obtained in the zeroth approximation [4, 5]. This system corresponds to a sharp decrease in the load at supercritical deformations. The relation between the load and the flexure has been found in [4] during a treatment of the first approximation and also in [5] by a variational method which uses the solution of the zeroth approximation as a coordinate function. The same results has been obtained previously [10] by a geometric method, which turns out to be equivalent to the method of the asymptotic integration of system (5.4) when constructing the internal boundary layer. The dependence $C^0(\gamma)$, which is presented at the end of Section 7, was obtained in [9].

Systems of non-linear equations of the type of (4.1) for the axisymmetrical deformation of shells of revolution have been considered in many papers ([11–13], etc). In [12, 13] and a number of other papers, it is recommended "with the aim of refinement" that the forces and moments should be divided by the unit of length of the median surface after deformation. This is quite possible if the elasticity relations are changed simultaneously. If, however, the same elasticity relations are used, then, as was shown in [7], a system is obtained to which no elastic energy corresponds. In the case of the problem on the limit load under axial compression, which is being considered here, the equations from the above-mentioned papers enable one to determine the value of C^0 in (6.3) correctly and should give an erroneous value of the correction C^1. It is interesting to note that the problem considered in [4, 5] concerning supercritical deformations, associated with a mirror reflection, imposes higher than usual requirements on the initial system. Calculations showed that the use of the system from [12, 13] does not allow one to determine even the leading term in the expression for the axial force correctly.

The limit value of the force P has been found above. Bifurcation into a non-axisymmetrical form of equilibrium may precede the limit load. In particular, this will always be so in the case of shells which have segments with a negative Gaussian curvature, because, in the case of bifurcation $P = O(Eh^2\mu^{2/3})$ for such shells (see [14]). The solution of system (5.4) can be used to solve the bifurcation problem.

This research was supported financially by the Russian Foundation for Basic Research (96-01-00411), the International Science Foundation and the Russian Government (NW5300).

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Translated by E.L.S.